dynamic pressure); 2) introducing damping into the ballast system (at least up to the point of critical damping for the ballast mounted on a stationary platform) produces the phase lag sooner in the flight than if there were no internal damping and therefore promotes earlier divergence of the oscillations; and 3) aerodynamic damping delays the onset of divergence.

The analysis is presented more completely, and results obtained are discussed more fully in Ref. 1.

Reference

[1] Malcolm, G. N., "The Effects of a Nonrigidly Supported Ballast on the Dynamics of a Slender Body Descending Through the Atmosphere," TN D-5622, 1970, NASA.

Characteristic Exponents for the Triangular Points in the Elliptic Restricted Problem of Three Bodies

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Danby[3] studied the linear stability of the triangular points numerically using Floquet theory. He presented transition curves that separate the stable from the unstable orbits in the μ-e plane (μ is the ratio of the smaller primary to the sum of the masses of the two primaries, and e is the eccentricity of the primaries' orbit). These curves intersect the μ axis at μ_a and μ_0, where $\mu_a = 0.03852$ is the limiting value of μ for stable orbits in the circular case, and $\mu_0 = 0.02859$ is the value of μ such that one of the periods of motion about the triangular points is exactly twice the period of the orbit of the primaries in the circular case.

Bennett[2] obtained a first-order complicated analytical expression for the transition curves μ_0 using an analytical technique for determination of characteristic exponents. Alfriend and Rand[1] obtained second-order analytical expressions for the transition curves at μ_a and μ_0 using the method of multiple scales.[5–7] Nayfeh and Kamel[8] determined fourth-order analytical expressions for the transition curves using a perturbation technique.

In this Note, we obtain a second-order analytical expression for the characteristic exponents and the transition curves using Floquet theory[9] and a perturbation technique used by Whittaker[10] in the treatment of Mathieu equation. We use the same formulation of Refs. 1 and 8.

The first variational equation about the triangular points can be transformed into

$$x'' - 2y' - gh_2x = 0 \tag{1}$$

$$y'' + 2x' - gh_1y = 0 \tag{2}$$

where primes denote differentiation with respect to the true anomaly f of the smaller body,

$$h_{1,2} = \tfrac{3}{2}\{1 \pm [1 - 3\mu(1 - \mu)]^{1/2}\} \tag{3}$$

$$g(e,f) = (1 + e\cos f)^{-1} \tag{4}$$

It is known from Floquet theory[9] that x and y have solutions of the form

$$x,y = e^{\gamma f}[\phi(f),\psi(f)] \tag{5}$$

where ϕ and ψ are periodic with periods of 2π or 4π. Moreover, the transition curves that separate the stable from unstable domains in the μ-e plane correspond to $\gamma = 0$. In the interval $0 \leq \mu < \mu_a$, where μ_a is the critical value above which the triangular points are unstable in the case $e = 0$, the period 2π corresponds to $\mu = 0$ whereas 4π corresponds to $\mu_0 = 0.02589$. Therefore, there are transition curves that intersect the μ axis at $\mu = 0$ and μ_0. In the former case, the transition curve is the e axis. In this note, we determine the characteristic exponents near μ_0 and then deduce the transition curves.

We seek expansions for ϕ, ψ, γ, and μ of the form

$$\phi = \phi_0 + e\phi_1 + e^2\phi_2 + \ldots \tag{6}$$

$$\psi = \psi_0 + e\psi_1 + e^2\psi_3 + \ldots \tag{7}$$

$$\gamma = e\gamma_1 + e^2\gamma_2 + \ldots \tag{8}$$

$$\mu = \mu_0 + e\mu_1 + e^2\mu_2 + \ldots \tag{9}$$

Substituting Eq. (9) into Eq. (3), and expanding for small e, we get

$$h_1 = \sum_{n=0}^{\infty} a_ne^n = a_0 + ea_1 + e^2a_2 + \ldots \tag{10}$$

$$h_2 = \sum_{n=0}^{\infty} b_ne^n = b_0 + eb_1 + e^2b_2 + \ldots \tag{11}$$

where

$$a_0 = \tfrac{3}{2}\{1 + [1 - 3\mu_0(1 - \mu_0)]^{1/2}\} \tag{12}$$

$$b_0 = \tfrac{3}{2}\{1 - [1 - 3\mu_0(1 - \mu_0)]^{1/2}\} \tag{13}$$

$$b_1 = -a_1 = (9/4\kappa)(1 - 2\mu_0)\mu_1 \tag{14}$$

$$b_2 = -a_2 = (9/4\kappa)((1 - 2\mu_0)\mu_2 - \{1 - \tfrac{3}{4}[(1 - 2\mu_0)/\kappa]^2\}\mu_1^2) \tag{15}$$

$$\kappa = [1 - 3\mu_0(1 - \mu_0)]^{1/2} \tag{16}$$

On substitution of Eqs. (5–11) into Eqs. (1) and (2), expansion for small e, and equating the coefficients of e^0, e, and e^2 to zero, we obtain

order e^0

$$\phi_0'' - 2\psi_0' - b_0\phi_0 = 0 \tag{17}$$

$$\psi_0'' + 2\phi_0' - a_0\psi_0 = 0 \tag{18}$$

order e

$$\phi_1'' - 2\psi_1' - b_0\phi_1 = -2\gamma_1\phi_0' + 2\gamma_1\psi_1 + b_1\phi_0 - b_0\phi_0\cos f \tag{19}$$

$$\psi_1'' + 2\phi_1' - a_0\psi_1 = -2\gamma_1\psi_0' - 2\gamma_1\phi_0 + a_1\psi_0 - a_0\psi_0\cos f \tag{20}$$

order e^2

$$\phi_2'' - 2\psi_2' - b_0\phi_2 = -2\gamma_2(\phi_0' - \psi_0) - 2\gamma_1(\phi_1' - \psi_1) - \gamma_1^2\phi_0 + b_2\phi_0 + b_1\phi_1 + b_0\phi_0\cos^2 f - (b_1\phi_0 + b_0\phi_1)\cos f \tag{21}$$

$$\psi_2'' + 2\phi_2' - a_0\psi_2 = -2\gamma_2(\psi_0' + \phi_0) - 2\gamma_1(\psi_1' + \phi_1) - \gamma_1^2\psi_0 + a_2\psi_0 + a_1\psi_1 + a_0\psi_0\cos^2 f - (a_1\psi_0 + a_0\psi_1)\cos f \tag{22}$$

The solution of Eqs. (17) and (18) is

$$\phi_0 = A\cos\tau + B\sin\tau \tag{23}$$

$$\psi_0 = \alpha B\cos\tau - \alpha A\sin\tau \tag{24}$$

where

$$\tau = f/2, \quad \alpha = (b_0 + 1/4) + (a_0 + 1/4)^{-1} \tag{25}$$

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This solution determines the right-hand sides of Eqs. (19) and (20). Thus,

$$\phi_1'' - 2\psi_1' - b_0\phi_1 = P_{11}\cos\tau + Q_{11}\sin\tau - \tfrac{1}{2}b_0A\cos 3\tau - \tfrac{1}{2}b_0B\sin 3\tau \quad (26)$$

$$\psi_1'' + 2\phi_1' - a_0\psi_1 = P_{12}\cos\tau + Q_{12}\sin\tau - \tfrac{1}{2}a_0\alpha B\cos 3\tau + \tfrac{1}{2}a_0\alpha A\sin 3\tau \quad (27)$$

where

$$P_{11} = \gamma_1(2\alpha - 1)B + (b_1 - b_0/2)A \quad (28a)$$

$$P_{12} = \gamma_1(\alpha - 2)A + \alpha(a_1 - a_0/2)B \quad (28b)$$

$$Q_{11} = -\gamma_1(2\alpha - 1)A + (b_1 + b_0/2)B \quad (28c)$$

$$Q_{12} = \gamma_1(\alpha - 2)B - \alpha(a_1 + a_0/2)A \quad (28d)$$

Since ϕ and ψ are periodic, the secular terms in the particular solution for ϕ_1 and ψ_1 must vanish. The conditions which must be satisfied for there to be no secular terms are

$$P_{11} = \alpha Q_{12}, \quad Q_{11} = -\alpha P_{12} \quad (29)$$

Then, the solution of Eqs. (26) and (27) is

$$\phi_1 = RA\cos 3\tau + RB\sin 3\tau \quad (30)$$

$$\psi_1 = -\alpha P_{12}\cos\tau - \alpha Q_{12}\sin\tau + SB\cos 3\tau - SA\sin 3\tau \quad (31)$$

where

$$R = [3\alpha a_0 + b_0(a_0 + \tfrac{9}{4})]/6$$
$$S = [3b_0 + \alpha a_0(b_0 + \tfrac{9}{4})]/6 \quad (32)$$

Substituting for the P's and Q's from Eq. (28) into (29), and rearranging, we get

$$[b_1 - b_0/2 + \alpha^2(a_1 + a_0/2)]A - \gamma_1 \times (1 - 4\alpha + \alpha^2)B = 0 \quad (33)$$

$$\gamma_1(1 - 4\alpha + \alpha^2)A + [b_1 + b_0/2 + \alpha^2(a_1 - a_0/2)]B = 0 \quad (34)$$

For a nontrivial solution,

$$\gamma_1^2 = -\frac{[b_1 - b_0/2 + \alpha^2(a_1 + a_0/2)][b_1 + b_0/2 + \alpha^2(a_1 - a_0/2)]}{(1 - 4\alpha + \alpha^2)^2} \quad (35)$$

Substituting for the zeroth- and first-order solutions into Eqs. (21) and (22) gives

$$\phi_2'' - 2\psi_2' - b_0\phi_2 = P_{21}\cos\tau + Q_{21}\sin\tau + \text{nonsecular producing terms} \quad (36)$$

$$\psi_2'' + 2\phi_2' - a_0\psi_2 = P_{22}\cos\tau + Q_{22}\sin\tau + \text{nonsecular producing terms} \quad (37)$$

where the P's and Q's are given in the appendix. For there to be no secular terms in ϕ_2 and ψ_2,

$$P_{21} = \alpha Q_{22}, \quad Q_{21} = -\alpha P_{22} \quad (38)$$

Substituting for the P's and Q's from the appendix and rearranging, we obtain

$$(b_2 - \xi_1)A - \eta B = 0, \quad \eta A + (b_2 - \xi_2)B = 0 \quad (39)$$

where

$$\xi_{1,2} = \{\pm 2(2\alpha^3 a_0 - 1 - \alpha^2)b_1 - 2(b_0 + \alpha^2 a_0) + 2(b_0R + \alpha a_0 S) + \alpha^3(4b_1^2 + a_0^2) + \gamma_1^2 \times (1 - 4\alpha + 5\alpha^2 - \alpha^3)\}/4(1 - \alpha^2) \quad (40)$$

$$\eta = [(1 - 4\alpha + \alpha^2)\gamma_2 - 2(2 - \alpha)\alpha^2\gamma_1 b_1]/2(1 - \alpha^2) \quad (41)$$

Since A/B is known from either Eq. (33) or Eq. (34), Eqs. (39) allow the determination of b_2 and η, and hence γ_2 as functions of μ_0, b_1, and γ_1; that is,

$$b_2 = (A^2\xi_1 + B^2\xi_2)/(A^2 + B^2), \quad \eta = AB(\xi_2 - \xi_1)/(A^2 + B^2) \quad (42)$$

To second approximation the transition curves are given by $\gamma_1 = \gamma_2 = 0$. With $\gamma_1 = 0$, Eq. (35) yields the following two values for b_1

$$b_1 = \pm(b_0 - \alpha^2 a_0)/2(1 - \alpha^2) \quad (43)$$

Equations (33) and (34) show that the positive sign in Eq. (43) corresponds to $B = 0$ while the negative sign corresponds to $A = 0$. Substituting into Eq. (42) either b_1 with the positive sign and $B = 0$ or b_1 with the negative sign with $A = 0$ gives

$$b_2 = \{2(2\alpha^3 a_0 - 1 - \alpha^3)|b_1| - 2(b_0 + \alpha^2 a_0) + 2(b_0R + \alpha a_0 S) + \alpha^3(4b_1^2 + a_0^2)/4(1 - \alpha^2)\} \quad (44)$$

Evaluating Eqs. (43) and (44) at $\mu = \mu_0 = 0.02589$ gives $\mu_1 = \pm 0.05641$ and $\mu_2 = 0.01504$, and hence the transition curves are given by

$$\mu = 0.02589 \pm 0.05641e + 0.01504e^2 + 0(e^3) \quad (45)$$

This expression for the transition curves in full agreement with those of Refs. 1 and 8.

Appendix A

$$P_{21} = [4b_2 - 2b_1 + 2b_0 - 2Rb_0 - \gamma_1^2(1 - 4\alpha + 2\alpha^2)] \times (A/4) + [\gamma_2(2\alpha - 1) - \gamma_1^2(2a_1 - a_0)](B/2) \quad (A1)$$

$$Q_{21} = [\gamma_2(1 - 2\alpha) + \gamma_1^2\alpha^2(2a_1 + a_0)](A/2) + [4b_2 + 2b_1 + 2b_0 - 2Rb_0 - \gamma_1^2(1 - 4\alpha + 2\alpha^2)](B/4) \quad (A2)$$

$$P_{22} = [\gamma_2(\alpha - 2) - \gamma_1\alpha a_0 - 2\gamma_1\alpha(\alpha - 1)a_1] \times (A/2) + [2\alpha(2a_2 - a_1 + a_0) - \alpha^2(2a_1 - a_0)^2 - 2Sa_0 \times \gamma_1^2\alpha(\alpha - 2)](B/4) \quad (A3)$$

$$Q_{22} = [-2\alpha(2a_2 + a_1 + a_0) + \alpha^2(2a_1 + a_0)^2 + 2a_0S - \gamma_1^2\alpha(\alpha - 3)](A/4) + [\gamma_2(\alpha - 2) + \gamma_1\alpha a_0 - 2\gamma_1\alpha(\alpha - 1)a_1](B/2) \quad (A4)$$

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